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FINITE GROUPS WITH SOME SUBGROUPS OF SYLOW SUBGROUPS *S*-SUPPLEMENTED

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Abstract

A subgroup H is called s-supplemented in G, if there exists a subgroup K of G such that $G = HK$ and $H \cap K \leq H_{SG}$, where H_{SG} is the largest subnormal subgroup of G contained in H. In this paper, we investigate the influence of s-supplemented primary subgroups in finite groups. Some new results about p-nilpotency of finite groups are obtained.

1. Introduction

Let H be a subgroup of G. Then, a subgroup K of G is called a *supplement* of H in G, if $G = HK$. It is of interest to study the structure

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of the group by using supplement of subgroups of finite groups. In [7] and [8], Kegel proved that a group G is soluble, if every maximal subgroup of G either has a cyclic supplement in G or if some nilpotent subgroup of G has a nilpotent supplement in G. On the other hand, by the well-known Hall's theorem [6], a group G is soluble, if and only if every Sylow subgroup of G has a complement in G. Recently, in [12, 13], Wang obtained some new characterizations for soluble and supersoluble groups by using some c-normal and c-supplemented subgroups.

In this paper, we remove the normal supplement condition and replace the c-supplement assumption with the s-supplement assumption for the subgroups of G. We obtain the p-nilpotency of G and the related results.

All the groups in this paper are finite. Most of the notation is standard and can be found in [4] and [11].

Definition 1.1. A subgroup H of G is called *s-supplemented* in G, if there exists a subgroup K of G such that $G = HK$ and $H \cap K \leq H_{SG}$, where H_{SG} is the largest subnormal subgroup of G contained in H. In this case, K is said to be an *s-supplement* of H in G.

Recall that a subgroup H of G is said to be *c-supplemented* in G, if there exists a subgroup K of G such that $G = HK$ and $H \cap K \leq H_G$ [13]. A subgroup H is said to be *s-normal* in G, if there exists a subnormal subgroup N of G such that $HN = G$ and $H \cap N \leq H_{SG}$ [14]. Hence, s-supplementation is a generalization of s-normality and c-supplementation. Moreover, we have s-supplementation cannot imply s-normality.

Example 1. $A_5 = C_5A_4$ and $C_5 \cap A_4 = 1$. Both C_5 and A_4 are c-supplemented in A_5 and so s-supplemented in A_5, but neither of them is s-normal in A_5, since A_5 is simple.

S-supplementation cannot imply c-supplementation.

Example 2. Let $G = Z_p \wr Z_q$, where Z_p and Z_q are the groups of prime p and q, respectively, $(p < q)$. Then, evidently, every subgroup H of G such that $H \cong Z_p$ is s-normal and so s-supplemented in G, but not c-supplemented in G.

2. Preliminaries

For the sake of convenience, we first list here some known results, which will be useful in the sequel.

Lemma 2.1. *Let G be a group. Then*:

(1) *If H is s-supplemented in G*, $H \leq M \leq G$, *then H is s-supplemented in M.*

(2) *Let* $N \trianglelefteq G$ *and* $N \leqslant H$. *Then H is s-supplemented in G, if and only if* H / N *is s-supplemented in* G / N.

(3) *Let* π *be a set of primes. Let N be a normal* π'-*subgroup and H be a* π-*subgroup of G. If H is s-supplemented in G, then* HN / N *is s-supplemented in* G / N. *If furthermore N normalizes H, then the converse also holds.*

Proof. The claims in (1)-(3) are easy exercises left to the reader.

Lemma 2.2 [10, Lemma 2.7]**.** *If* $L \lhd\lhd G$ *and L is a p-subgroup, then* $L \leq O_p(G)$.

Lemma 2.3. *Let* π *be a set of prime divisor of* $|G|$. *If* $G \in E_\pi$, *then every subnormal subgroup and every composition factor of G belongs to* E_π.

Proof. It is clear that every normal subgroup of G belongs to E_π when $G \in E_\pi$. For every subnormal subgroup K of G, there exists a subnormal series of G

$$K = K_0 \trianglelefteq K_1 \trianglelefteq \cdots \trianglelefteq K_{n-1} \trianglelefteq K_n = G.$$

Since $G \in E_\pi$, then $K_{n-1} \in E_\pi$ and K belongs to E_π by the induction. On the other hand, it is easy to know every quotient group of G belongs to E_π when $G \in E_\pi$. Similarly, every composition factor belongs to E_π by the induction. This completes the proof.

Lemma 2.4. *Let G be a finite group and P be a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ such that $(|G|, p-1) = 1$. Suppose that there exists a maximal subgroup P_1 of P such that P_1 is s-supplemented in G. Then G is not a non-abelian simple group and $G \in D_{p'}$.*

Proof. (1) $G \in D_{p'}$.

We prove this by induction on the order of G. Since P_1 is s-supplemented in G, there exists a subgroup K of G such that $P_1 K = G$ and $P_1 \cap K \leq (P_1)_{SG}$.

If $P_1 \cap K = 1$, then $|K|_p = p$. Let K_p denote a Sylow p-subgroup of K. Then $N_K(K_p)/C_K(K_p)$ is isomorphic to a subgroup of $Aut(K_p)$. Hence, the order of $N_K(K_p)/C_K(K_p)$ must divide $(|G|, p-1) = 1$. Therefore, $N_K(K_p) = C_K(K_p)$ by Burnside's p-nilpotent theorem and hence K is p-nilpotent. It is clear that the normal p-complement $K_{p'}$ is a Hall p'- subgroup of G and hence $G \in E_{p'}$. If p is an odd prime, then G is soluble and hence $G \in D_{p'}$. If $p = 2$, then [3, Main Theorem] implies that $G \in C_{2'}$. By [1, P.547], if π is a set of odd primes and G satisfies E_π and $E_{\pi'}$, then $G \in D_{\pi'}$. Hence we have that $G \in D_{2'}$.

If $P_1 \cap K \neq 1$ and $K < G$, then $P_1 \cap K = (P_1)_{SG} \cap K \lhd\lhd K$. It is easy to see that $P_1 \cap K$ is s-supplemented in K. Since $|P \cap K : P_1 \cap K| = |P : P_1| = p$, by the hypotheses, we have that $K \in D_{p'}$. With the similar argument, we have $G \in D_{p'}$.

Now, we may assume $P_1 \cap K \neq 1$ and $K = G$, i.e., $P_1 \lhd\lhd G$. If p is an odd prime, then G is soluble since $(|G|, p-1) = 1$ and hence $G \in D_{p'}$. If $p = 2$, then there exists a subnormal series of G such that

$P_1 \trianglelefteq M_1 \trianglelefteq M_2 \trianglelefteq \cdots \trianglelefteq M_n = G$. It is easy to see that $|M_1/P_1| = 2n_1$ or n_2, where n_1 and n_2 are both odd numbers. Now, we have M_1 is soluble. By the same argument, we obtain that G is soluble. Therefore $G \in D_{p'}$.

(2) G is not a non-abelian simple group.

Assume that G is a non-abelian simple group. By assumption, there exists a subgroup K of G such that $G = P_1K$ and $P_1 \cap K \leq (P_1)_{SG} = 1$. In particular, $|G : K| = p^{\alpha}$, $\alpha \geq 1$. By [5, Theorem 1], we know that either K is a Hall p'-subgroup of G or G is isomorphic to $PSU_4(2) \cong PSP_4(3)$, and K is the parabolic subgroup of index 27 or G is isomorphic to A_n with $5 \leqslant n = p^r$, $r \geq 2$ and $K \cong A_{n-1}$. Clearly, K is not a Hall p'-subgroup of G since $|G : K| = |P_1 : P_1 \cap K| \leq |P_1| < |P|$. If $G \cong PSU_4(2)$, then $|G| = 2^6 \cdot 3^4 \cdot 5$ and $|K| = 2^6 \cdot 3 \cdot 5$. By (1) and the condition, $G \in E_{p'}$ and there exists a Hall p'-subgroup $G_{p'}$ of G such that $|G_{p'}| = 2^6 \cdot 5$. Hence, we have $|G : G_{p'}| = 3^4$, contrary to [5, Theorem 1]. But in the last case, $|P_1| = n = p^r$, $(n!/2) = (12 \ldots p^r)/2$. If $r > 1$, then $p^2 \mid |A_{n-1}|$ and $p^2 \mid |P : P_1|$. Therefore, G is not a non-abelian simple group.

The theorem is proved.

3. Main Results

Theorem 3.1. *Let G be a finite group and P be a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Suppose that every maximal subgroup of P is s-supplemented in G, then $G/O_p(G)$ is p-nilpotent.*

Proof. Assume that the theorem is false and choose G to be a counterexample of smallest order. By Lemma 2.4, we have $G \in E_{p'}$. Furthermore, we have

(1) $O_p(G) = 1$.

If $O_p(G) = P$, then $G / O_p(G)$ is a p'-group and of course, it is p-nilpotent, a contradiction. If $1 \leq O_p(G) < P$, then $G / O_p(G)$ satisfies the hypotheses and the minimal choice of G implies that $G / O_p(G) \cong G / O_p(G) / O_p(G / O_p(G))$ is p-nilpotent, a contradiction.

(2) For every maximal subgroup P_1 of P, the s-supplement of P_1 is p-nilpotent.

Let P_1 be a maximal subgroup of P. By hypotheses, P_1 is s-supplemented in G. So, there exists a subgroup K_1 of G such that $G = P_1 K_1$ and $P_1 \cap K_1 \leq (P_1)_{SG}$. By Lemma 2.2, we have that $P_1 \cap K \leq (P_1)_{SG} \leq O_p(G) = 1$. Now $|K_1|_p = p$. Let K_{1p} denote the Sylow p-subgroup of K_1. Then, $N_{K_1}(K_{1p}) / C_{K_1}(K_{1p})$ is isomorphic to a subgroup of $Aut(K_{1p})$. Hence, the order of $N_{K_1}(K_{1p}) / C_{K_1}(K_{1p})$ must divide $(|G|, p-1) = 1$. Therefore $N_{K_1}(K_{1p}) = C_{K_1}(K_{1p})$. Burnside's p-nilpotent theorem [11, 10.1.8] implies that K_1 is p-nilpotent.

(3) G is p-nilpotent.

Let P_1 be a maximal subgroup of P. By (1) and (2), there exists a p-nilpotent subgroup K_1 of G such that $G = P_1 K_1$. Let $K_1 = K_{1_p} K_{1_{p'}}$ and $N = N_G(K_{1_{p'}})$. Clearly, $K_1 \leq N$ and $G = PN$. If $P \leq N$, then $N = G$, a contradiction. So, we may assume that $P \cap N < P$. There exists a maximal subgroup P_2 of P such that $P \cap N \leq P_2$. By hypotheses, P_2 is s-supplemented in G. (2) indicates that the supplement K_2 of P_2 is p-nilpotent. We denote $K_2 = K_{2_p} K_{2_{p'}}$. Now both $K_{1_{p'}}$ and $K_{2_{p'}}$ are Hall p'-subgroup of G. Since $(|G|, p-1) = 1$, by Lemma 2.4, we have $G \in D_{p'}$, these two subgroups are conjugate in G. Say $K_{1_{p'}} = (K_{2_{p'}})^g$. Since $G = P_2 K_2$ and $K_{2_{p'}} \trianglelefteq K_2$, we may choose $g \in P_2$. K_2^g normalizes

$K_{2_{p'}}^{g} = K_{1_{p'}}$ and hence $K_2^g \leq N$. Now $G = (P_2K_2)^g = P_2N$. Therefore, $P = P \cap G = P_2(P \cap N) \leq P_2$. Since $P \cap N \leq P_2$, we have that $P \leq P_2$, a contradiction.

Based on the discussion as above and [2], $G / O_p(G)$ is p-nilpotent.

Lemma 3.2 [9, Lemma 2.4]**.** *Let G be a finite group and p be a prime divisor of* $|G|$ *such that* $(|G|, p^2 - 1) = 1$. *Assume that the order of G is not divisible by* p^3. *Then G is p-nilpotent.*

Theorem 3.3. *Let G be a finite group and p be a prime divisor of* $|G|$ *such that* $(|G|, p^2 - 1) = 1$. *Assume that every second maximal subgroup of the Sylow p-subgroup of P is s-supplemented in G, then* $G / O_p(G)$ *is soluble p-nilpotent.*

Proof. Assume that the claim is false and choose G to be a counterexample of minimal order. Furthermore, we have

(1) $O_p(G) = 1$.

If $O_p(G) = P$, then $G / O_p(G)$ is a p'-group and of course, it is p-nilpotent, a contradiction. If $O_p(G) = P_1$, where P_1 is the maximal subgroup of P, then $G / O_p(G)$ is p-nilpotent since $(|G|, p - 1) = 1$ and $|G / O_p(G)|_p = p$, a contradiction. If $O_p(G) = P_2$, where P_2 is the second maximal subgroup of P, then $p^3 \nmid |G / O_p(G)|$. Hence, $G / O_p(G)$ is p-nilpotent by Lemma 3.2. If $1 < O_p(G) < P_2$, then $G / O_p(G)$ satisfies the hypotheses and the minimal choice of G implies that $G / O_p(G) \cong G / O_p(G) / O_p(G / O_p(G))$ is p-nilpotent, a contradiction.

(2) $|G|$ is divisible by p^3.

If $p^3 \nmid |G|$, then G is p-nilpotent by Lemma 3.2, a contradiction.

(3) For every second maximal subgroup P_1 of a Sylow subgroup P of G, the s-supplement of P_1 is p-nilpotent.

Let P be a Sylow p-subgroup of G and P_1 be a second maximal subgroup of P. By hypotheses, P_1 is s-supplemented in G. So, there exists a subgroup K_1 of G such that $G = P_1K_1$ and $P_1 \cap K_1 \leq (P_1)_{SG}$. By Lemma 2.2, we have $P_1 \cap K_1 \leq (P_1)_{SG} \leq O_p(G) = 1$. Now $|K_1|_p = p^2$. By hypotheses and Lemma 3.2, we have K_1 is p-nilpotent.

(4) G is p-nilpotent.

Let $N = N_G(K_{1_{p'}})$ and $K_1 = K_{1_p}K_{1_{p'}}$. By (3), $K_1 \leq N$. So, we have $G = P_1K_1 = P_1N$. If $N = G$, then G is p-nilpotent, a contradiction. Let $P_1 \leq \overline{P_1} \leq P$, where $\overline{P_1}$ is a maximal subgroup of Sylow subgroup P of G. Hence, $G = P_1K_1 = \overline{P_1}K_1 = \overline{P_1}N$. If $\overline{P_1} \leq N$, then G is p-nilpotent, a contradiction. So, we may assume $\overline{P_1} \cap N < \overline{P_1}$. We may choose a maximal subgroup P_2 of $\overline{P_1}$ such that $\overline{P_1} \cap N \leq P_2$. It is clear that P_2 is the second maximal subgroup of P. By (3), P_2 is s-supplemented in G and the supplement K_2 of P_2 is p-nilpotent. We denote $K_2 = K_{2_p}K_{2_{p'}}$. Since $(|G|, p^2 - 1) = 1$, [3, Main Theorem] or the odd order theorem [2] implies that $G \in C_{p'}$. Now both $K_{1_{p'}}$ and $K_{2_{p'}}$ are Hall p'- subgroup of G, these two subgroups are conjugate in G. Let $K_{1_{p'}} = (K_{2_{p'}})^g$. Since $G = P_2K_2$ and $K_2 \leq N_G(K_{2_{p'}})$, we may choose $g \in P_2$. ${K_2}^g$ normalizes ${K_{2_{p'}}}^g = K_{1_{p'}}$ and hence ${K_2}^g \leq N$. Now $G = (P_2K_2)^g = P_2N$. Therefore $\overline{P_1} = \overline{P_1} \cap P_2N = P_2(\overline{P_1} \cap N) = P_2$, contrary to the condition.

The final contradiction completes our proof.

Theorem 3.4. *Let G be a finite group. Then G is soluble, if and only if every Sylow subgroup of G is s-supplemented in G.*

Proof. If G is soluble, then by [6, Main Theorem], every Sylow subgroup of G is complemented in G. It is clear that every Sylow subgroup of G is s-supplemented in G.

Conversely, assume that every Sylow subgroup P of G is s-supplemented in G. By [6, Main Theorem], we only need to prove that P is complemented in G. Let K be an s-supplement of P in G. Then $G = PK$ and $P \cap K \leq P_{SG}$.

If $P \cap K = 1$, then P is complemented in G.

If $P \cap K \neq 1$, then $P \cap K = P_{SG} \cap K \lhd\lhd K$. Note that $|G|_p = |P|(|K|_p / |P_{SG} \cap K|)$, hence $|K|_p = |P_{SG} \cap K|$ and $P \cap K = P_{SG} \cap K \trianglelefteq K$. By the Schur-Zassenhaus theorem [11, Theorem 9.1.10], we have that $K = (P \cap K)K_{p'}$, where $K_{p'}$ is the Hall p'-subgroup of K. Now, $G = PK = P(P \cap K)K_{p'} = PK_{p'}$ and $P \cap K_{p'} = 1$. Therefore, P is complemented in G. The theorem is proved.

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